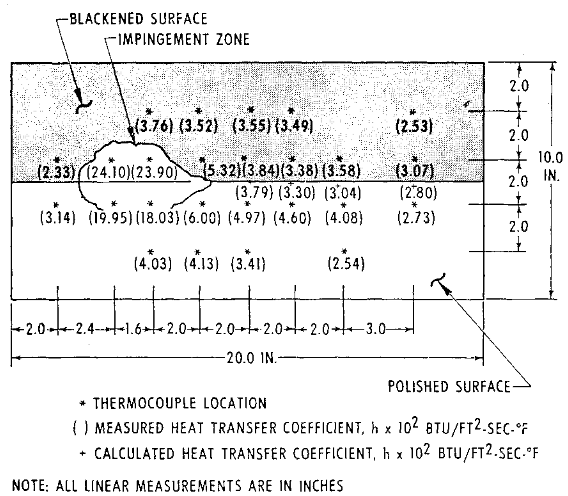

Fig. 4 Map of heat-transfer coefficients.

copper plate. Also evident were general flow lines and patterns (see Fig. 3). As would be expected, the area directly in line with the nozzle received severe deposition, but areas to the side and downstream received much less. The areas upstream of impingement received slight or no deposition. A map of the calculated and measured heat-transfer coefficients is presented in Fig. 4.

Turbulent flat-plate heat-transfer coefficients were estimated using Eq. (3) and the following expressions and assumptions to calculate gas properties:

$$C_p = \gamma R/(\gamma - 1)$$

$$P_R = 4\gamma/(9\gamma - 5)$$

$$\mu_R = 46.6 \times 10^{-10}(M)^{1/2}(T)_R{}^6 \text{ lb/in.-sec} \qquad (3a)$$

The quantity ρu was approximated from the average mass flow from the igniter and the jet cross-sectional area immediately before impingement. The quantity $(T_{og} - T_p)$ for the experimental conditions was taken to be approximately 5000°F. As can be seen, the experimental values are slightly higher than the analytical, which may be due to aluminum oxide solidifying on the surface, or possibly the quantity $(T_{og} - T_p)$ is less than the approximated 5000°F. However, the turbulent flat-plate theory does predict the correct level of the heat-transfer coefficient.

The effects of radiation were most visible in the impingement zone. In this region, radiation was approximately 20% of the total heat transfer; outside this area, the contribution was indiscernible. At some locations, the thermocouples behind the painted surface indicated lower over-all heat transfer than those behind the polished surface, possibly because of the insulating effect of the paint.

Heat transfer in the impingement zone was 500 to 600% higher than a turbulent boundary-layer analysis would predict; this was much higher than the single-phase data of Ref. 6, which indicated an increase of 20%. Therefore, the two-phase nature of the jet (i.e., the alumina particles) probably are responsible for the large increase in heat transfer.

After firing, the deposited aluminum oxide was scraped from the direct impingement zone and weighed. The weight deposited divided by the deposition area and the firing duration gave a deposition rate equal to approximately 1.5% of the total particle flow rate.

Conclusions

Stagnation heat transfer, solidification of alumina particles, and radiation were the important modes of heat transfer in the impingement zone. Outside the impingement area, radiation and solidification were relatively unimportant; therefore, the greatest heat transferred was from convection, and turbulent boundary-layer theory is applicable. Although the actual heat transferred in a short flow region following the impingement zone was higher than indicated by turbulent boundary-layer theory, the theory can provide a fair estimate of the actual heat transferred.

References

[1] Shirnow, V. A., Verevochkin, E. E., and Brdlick, P. H., "Heat transfer between a jet and a held plate normal to flow," Intern. J. Heat Mass Transfer, 2 1–7 (1961).

[2] Myers, G. E., Schauer, J. J., and Eustis, R. H., "The plane turbulent wall jet," Parts 1 and 2, Dept. of Mechanical Engineering, Stanford Univ. TR 1 and 2, National Science Foundation Grant G9705 (1961).

[3] Metzer, D. E., "Spot cooling and heating of surfaces with impinging air jets," Dept. of Mechanical Engineering, Stanford Univ. TR 52, Contract NONR 225(23) (April 1962).

[4] Bauer, R. C. and Schlumpf, R. L., "Experimental investigation of free jet impingement on a flat plate," Arnold Engineering Development Center, AEDC TN 60-223 (March 1961).

[5] Stitt, L. E., "Interaction of highly underexpanded jets with simulated lunar surfaces," NASA TN-D-1095 (December 1961).

[6] Huang, G. C., "Investigations of heat transfer coefficients for air flow through round jets impinging normal to a heat transfer surface," American Society of Mechanical Engineers, Paper 62-HT-31 (1962).

Buckling of Axially Compressed Cylindrical Shells Reinforced by Circumferential Fiber Windings

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1. Introduction

METAL cylinders designed to withstand internal pressure are sometimes prestressed by circumferential filament windings, so that the axial and circumferential stresses in the cylinders are approximately equalized when the internal pressure acts. The lateral deflections and the initial stresses caused by the windings may conceivably reduce the capacity of such a cylinder to withstand an axial compressive force. Apparently, the windings have little effect on the elastic stability if they are bonded to the cylinder, since they cannot affect the net tractions N_x, N_θ, $N_{x\theta}$ in the composite structure. However, the situation is different if the fibers are free to slip on the cylinder when buckling occurs. This case is investigated for isotropic elastic cylinders in the present analysis. The infinitesimal theory of buckling is used. It is known that this theory gives axial buckling loads of cylinders higher than those observed in tests. Nevertheless, the infinitesimal theory yields an indication of the relative importance of the windings on the stability of the shell.

We suppose that there are no end constraints when the windings are applied. The effect of the windings is to cause a constant radial deflection w_0. After the shell is wound, end plates are inserted, and the shell is subjected to axial com-

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pression. Because of the end diaphragms, w_0 is not changed by the axial compression. We suppose that these diaphragms have negligible flexural stiffness, but they prevent radial and circumferential displacements at the ends.

The wound cylinder is regarded as a conservative mechanical system. Since the system is treated as a unit by means of energy principles, the force of the fibers on the shell need not be considered explicitly. It is supposed that the fibers do not adhere to the shell. It is assumed that the tension stress s_0 in the fibers during winding is constant. In practice, this condition is not realized, but an average tensile stress s_0 may be used.

2. General Equations for Cylindrical Shells

Consider an elastic cylindrical shell of length L, thickness h, and mean radius a. Let (u, v, w) be the axial, circumferential, and radial displacements of a point on the middle surface (positive w outward). The configuration $u = v = w = 0$ is understood to be the completely unstrained state. Let x be distance along the axis of the cylinder, measured from one end, and let θ be the angular cylindrical coordinate. The strain components of the middle surface are $(\epsilon_x, \epsilon_\theta, \gamma_{x\theta})$; and the incremental curvatures due to bending are κ_x, κ_θ, and $\kappa_{x\theta}$. The strains will be linearized in u, and the incremental curvatures will be linearized in (u, v, w). With these approximations, the following formulas are obtained[1-3]:

$$M = (v_\theta + w)/a \qquad N = (w_\theta - v)/a \tag{1}$$

$$\epsilon_x = u_x + \tfrac{1}{2}(v_x^2 + w_x^2) \qquad \epsilon_\theta = M + \tfrac{1}{2}(M^2 + N^2)$$
$$\gamma_{x\theta} = v_x + (u_\theta/a) + Mv_x + Nw_x \tag{2}$$

$$\kappa_x = -w_{xx} \qquad a\kappa_\theta = -(w_{\theta\theta} + w)/a$$
$$a\kappa_{x\theta} = -2w_{x\theta} \tag{3}$$

The membrane energy and the bending energy are

$$U_m = \frac{Gah}{1-\nu}\iint [\epsilon_x^2 + \epsilon_\theta^2 + 2\nu\epsilon_x\epsilon_\theta + \tfrac{1}{2}(1-\nu)\gamma_{x\theta}^2]dxd\theta \tag{4}$$

$$U_b = \frac{Gah^3}{12(1-\nu)}\iint [\kappa_x^2 + \kappa_\theta^2 + 2\nu\kappa_x\kappa_\theta + \tfrac{1}{2}(1-\nu)\kappa_{x\theta}^2]dxd\theta \tag{5}$$

By taking the proper origin for θ, we may make u and w even functions of θ, whereas v becomes an odd function of θ. Hence, the integrals in Eqs. (4) and (5) may be restricted to the range $0 \leqq \theta \leqq \pi$, if they are then doubled.

3. Reduction to Finite Degrees of Freedom

For a cylinder of infinite length, the displacement pattern accompanying buckling is doubly periodic. The fundamental region is a rhombus; it has subtended angle $2\pi/n$ from the axis of the cylinder, where n is a positive integer. Accordingly, (u_x, v, w) may be represented by double Fourier series. In the infinitesimal theory of buckling, the first terms of these series suffice. For a cylinder of finite length, it is commonly assumed that the number of lobes in the length is an integer m. To simplify the algebra, we let the length of the cylinder be π. This condition does not restrict the generality, since any unit of length might be used. To satisfy the conditions that the end diaphragms suppress the radial and circumferential displacements, we introduce the factor $\sin x$ in the formulas for v and w. Furthermore, we add w_0 to represent the initial displacement caused by the windings. Thus,

$$\begin{aligned} u &= u_0 x + u_2 \cos n\theta \sin 2mx \\ v &= v_2 \sin n\theta \sin x \cos 2mx \\ w &= w_0 + w_1 \sin x + w_2 \cos n\theta \sin x \cos 2mx \end{aligned} \tag{6}$$

The coefficients u_0, u_2, v_2, w_0, w_1, and w_2 are constants. The term u_0x accounts for initial elongation due to windings (Poisson-ratio effect). Also, it accounts for prebuckling contraction under the axial load and any supplemental uniform axial strain that accompanies buckling. By elementary theory,

$$w_0 = -(aAs_0/Eh) = -(aN_0/Eh) \tag{7}$$

where w_0 represents the deformation due to the initial winding process, s_0 is the tensile stress in the fibers during winding (assumed constant), E is Young's modulus of the shell, and A is the area of a unit length of a longitudinal cross section of the fibers. If there were no spaces between contacting fibers, A would be the thickness of the layer of fibers. The hoop compression in the shell (lb/in.) due to the winding is $N_0 = As_0$.

4. Evaluation of Membrane Energy

If Eq. (6) is substituted into Eq. (4) and the integrations are performed, the membrane energy U_m is expressed as a fourth-degree polynomial in u_0, u_2, v_2, w_0, w_1, w_2. However, fourth-degree terms are negligible in the infinitesimal theory of buckling; consequently, they may be dropped. For brevity, the following notations are introduced:

$$\alpha = (nv_2 + w_2)/a \qquad \beta = (v_2 + nw_2)/a \tag{8}$$

The details of the calculation of U_m are lengthy but routine. For that reason the final result is not given here.

5. Evaluation of Bending Energy

By Eqs. (3) and (6), the quantities κ_x, κ_y, and κ_{xy} are expressed in terms of w_0, w_1, w_2, x, θ. Hence, by Eq. (5), the strain energy of bending U_b is expressed in terms of w_0, w_1, w_2. As for U_m the calculation is lengthy but routine, and the final result is not given here.

6. Strain Energy of Fibers

The initial strain of the fibers is s_0/E. After the shell is wrapped, the fibers experience additional strain because of the effect of the axial load. The small displacements considered in the infinitesimal theory of buckling can cause no separation between the fibers and the shell, since a change of curvature greater than $1/a$ would be required to produce concave regions in the buckled cross section.

The increment in the circumference of the shell due to the deformation caused by the axial load is

$$\Delta c = -2\pi a + \int_0^{2\pi} [a^2 + v_\theta^2 + v^2 + \bar{w}_\theta^2 + \bar{w}^2 + 2av_\theta + 2a\bar{w} + 2v_\theta\bar{w} - 2v\bar{w}_\theta]^{1/2}\, d\theta$$

where, by Eq. (6), $\bar{w} = w_1 \sin x + w_2 H \cos n\theta$, and $H = \sin x \cos 2mx$. Here $\bar{w}$ denotes the part of w that results from the axial load. Expanding by the binomial series as far as quadratic terms, introducing Eq. (6), and integrating, we obtain, with Eq. (8),

$$\Delta c = 2\pi w_1 \sin x + (\pi a/2)H^2\beta^2$$

The incremental strain of a fiber due to the supplemental deformation is $\Delta c/2\pi a$. Consequently, the total strain of a fiber is

$$\epsilon = (w_1/a)\sin x + \tfrac{1}{4}H^2\beta^2 + (s_0/E) \tag{9}$$

The strain energy of all the fibers is

$$U_f = \pi a\bar{E}A \int_0^\pi \epsilon^2 dx \tag{10}$$

Consequently, if a fourth-degree term and an irrelevant constant term are dropped, substitution of Eq. (9) into Eq. (10) yields, after integration, U_f as a third-degree polynomial in terms of w_1 and β.

7. Potential Energy of External Forces

The mean extension of the shell is

$$e = \frac{1}{\pi}\int_0^{\pi} [u(L) - u(0)]d\theta$$

By Eq. (6), $u(0) = 0$. Also, since $L = \pi$, $u(L) = \pi u_0$. Consequently, $e = \pi u_0$. A part of the elongation (namely, $-\pi\nu w_0/a$) results from the initial deformation caused by the windings. Consequently, the mean extension that results from the axial load is $\pi(u_0 + \nu w_0/a)$.

If internal pressure p is present, a part of the axial load results from p. However, this effect will be treated separately, so that the net axial compression force is $\bar{F} = F - \pi a^2 p$. Accordingly, the potential energy of the thrust load is $\Omega_F = \pi(F - \pi a^2 p)(u_0 + \nu w_0/a)$.

Furthermore, there is potential energy due to the internal pressure p on the lateral surface. Its value is[3]

$$\Omega_p = \int_0^L dx \int_0^{\pi} [-2paw(1 + M + xM_x) + 2pav(N + xN_x)]d\theta$$

where (M, N) are given by Eq. (1). The total potential energy of external forces is $\Omega = \Omega_F + \Omega_p$. With v and w given by Eq. (6), the foregoing formulas express Ω as a second-degree polynomial in terms of w_0, w_1, w_2, u_0, and v_2.

8. Elimination of u_0 and u_2

The total potential energy is $V = U_m + U_b + U_f + \Omega$. The terms in this sum are given by the preceding analysis. The quantities u_0 and u_2 may be eliminated by the principle of stationary potential energy, $\partial V/\partial u_0 = \partial V/\partial u_2 = 0$, since these conditions provide separated linear algebraic equations for u_0 and u_2. Then V becomes a cubic polynomial in v_2, w_0, w_1, and w_2.

9. Deformation before Buckling

Before buckling, $v_2 = w_2 = 0$. Also, w_0 may be eliminated by Eq. (7). Then, if third-degree terms are dropped, the condition of stationary potential energy ($dV/dw_1 = 0$) yields the prebuckling value of w_1; namely,

$$w_1 = \frac{\nu(1 - \nu)(F - \pi a^2 p) + 2(1 - \nu)\pi a^2 p + [\pi h^2 a A s_0/6(1 + \nu)][(1/a^2) - \nu]}{Gh[\pi^2 - 8\nu^2 + (\pi^2 h^2/12a^2)(a^4 - 2\nu a^2 + 1) + (\pi^2 \bar{E} A/Eh)(1 - \nu^2)]} \quad (11)$$

where nonlinear terms in p and F have been discarded. The term w_1 represents the deformation before buckling due to axial load F and internal pressure p.

10. Infinitesimal Theory of Buckling

The preceding results yield a third-degree polynomial in v_2, w_1, w_2, and w_0 for the potential energy V of the slightly buckled shell. The linear terms in V do not contribute to the second variation of the potential energy. The third-degree terms that do not contain w_1 or w_0 contribute nothing to the second variation, since v_2 and w_2 vanish for the unbuckled form. It is to be noted that w_0 is a constant; its variation is zero.

We now replace v_2, w_1, and w_2, by $v_2 + \xi_1$, $w_1 + \xi_2$, $w_2 + \xi_3$, where (ξ_1, ξ_2, ξ_3) are arbitrary increments (variations). Then, V is transformed into $V + \Delta V$. The part of ΔV that is quadratic in the variables ξ_1, ξ_2, and ξ_3 is $\frac{1}{2}\delta^2 V$, where $\delta^2 V$ denotes the second variation of V. Hence,

$$\frac{1}{2}\delta^2 V = \sum_{i=1}^{3}\sum_{j=1}^{3} P_{ij}\xi_i\xi_j \quad (12)$$

The coefficients P_{ij} are symmetrical; that is, $P_{ij} = P_{ji}$. The buckling criterion is $\det(P_{ij}) = 0$, where the symbol "det" denotes "determinant." It happens that $P_{12} = P_{23} = 0$. Consequently the equation $\det(P_{ij}) = 0$ reduces to $P_{22}(P_{11}P_{33} - P_{13}^2) = 0$. Since P_{22} does not depend on m or n, the buckling load obtained from $P_{22} = 0$ represents the axially symmetrical form. Therefore, for thin shells, the Euler buckling load is determined by $P_{11}P_{33} = P_{13}^2$. The expressions for P_{11}, P_{33}, and P_{13} are obtained from the formula for V. They are the coefficients of ξ_1^2, ξ_3^2, and $2\xi_1$, ξ_3, respectively, in Eq. (12).

Table 1 Effect of hoop windings on buckling stress of a shell ($p = 0$)[a]

s_0	m	n	σ_{cr}/E
0	8	8	0.00377
1,000	8	8	0.00374
35,000	8	8	0.00374
70,000	8	8	0.00373

[a] L = 25 in., a = 10 in., h = 0.060 in., A = 0.05 in., $E = 30 \times 10^6$ lb/in.², $\bar{E} = 12 \times 10^6$ lb/in.², ν = 0.30.

11. Numerical Investigation

To apply the preceding theory, we recall that the length of the shell has been set equal to π. Hence, when a, h and A are specified, the geometric properties of the fiber-wound shell are determined. The material properties are given by E, $\bar{E}$ and ν.

If s_0, a, h, A, ν, $\bar{E}$, E, and p are specified, the deflection parameters w_0 and w_1 before buckling are determined by Eqs. (7) and (11), respectively. We note that w_1 is a linear function of $\bar{F}/G$, where $\bar{F} = F - \pi a^2 p$. Hence, if m and n are estimated, P_{11}, P_{33}, and P_{13} are expressed as linear functions of $\bar{F}/G$ by the theory of Sec. 10. Then, $\bar{F}/G$ is determined by the condition $P_{11}P_{33} = P_{13}^2$. The computations must be repeated with various integers m and n, until the values are found which make $\bar{F}/G$ a minimum. When $\bar{F}/G$ is determined, the buckling stress $\sigma_x = \sigma_{cr}$ is determined by the relation $\sigma_{cr} = \bar{F}/(2\pi ah)$.

To study the relative importance of hoop fiber windings, a steel cylinder was considered, with the following geometrical and physical parameters: L = 25 in., h = 0.060 in., a = 10 in., $E = 30 \times 10^6$ lb/in.², $\bar{E} = 12 \times 10^6$ lb/in.², $\nu = 0.30$ and A = 0.050 in. Since $L = \pi$ in the theory it is necessary to scale the length dimensions. For example, in the preceding theory, we must set $h = 0.060\ (\pi/25)$, $a = 10\ (\pi/25)$, and $A = 0.05\ (\pi/25)$. Values of m, n and σ_{cr}/E are given in Table 1, for a range of values of s_0.

12. Discussion of Results and Conclusions

Table 1 shows that m, n, and σ_{cr} are practically independent of the fiber tensions s_0, up to values of s_0 = 70,000 lb/in.² in the present example. At s_0 = 70,000 lb/in.², the compressive hoop stress in the steel cylinder is 58,300 lb/in.². At least for cylinders with approximately the values of a/h and A/h under discussion, there is little interaction between stresses caused by hoop windings and axial buckling stress.

The value 0.00377 for $s_0 = 0$, which was obtained by the energy method, is about the same as that derived by Timoshenko (namely, 0.00363) by consideration of equilibrium equations for the slightly buckled shell.[4] Hence, the relative importance of hoop windings is believed to be reasonably well estimated.

References

[1] Love, A. E. H., *Mathematical Theory of Elasticity* (Dover Publications, New York, 1944), 4th Ed.

[2] Novozhilov, V. V., *Foundations of the Nonlinear Theory of Elasticity* (Graylock Press, Rochester, New York, 1953).

[3] Langhaar, H. L. and Boresi, A. P., "Buckling of a cylindrical shell subjected to external pressure," Oster. Ingr.-Arch. **XIV**, 189–203 (1960).

[4] Timoshenko, S., *Theory of Elastic Stability* (McGraw-Hill Book Company, Inc., New York, 1936), p. 457.